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THREE-DIMENSIONAL ANALOG OF PRANDTL–MEYER WAVES

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This paper studies the regular partially invariant solution of the equations of gas dynamics which extends the Prandtl–Meyer solutions to the three-dimensional case. All singular manifolds of the third-order dynamic system that defines the solution are found, and its compactification is constructed.

Key words: *equations of gas dynamics, dynamic system, singular manifolds.*

Group-theoretical methods allow one to find extensive classes of exact solutions of the nonlinear mathematical models describing the three-dimensional motion of a medium. The obtained submodels are solved invoking methods of the theory of dynamic systems. The present paper deals with constructing an exact solution of the equations of gas dynamics that extends the classical Prandtl–Meyer solution to the three-dimensional case. This solution does not reduce to a simple wave and describes three-dimensional gas motion.

1. Derivation of the Dynamic System. The subalgebra $L_{4.23}$ of the symmetry algebra of the equations of gas dynamics (EGD) with the gas equation of state of general form [1]

$$L_{4.23} = \{\partial_x, t\partial_x + \partial_u, \partial_t, t\partial_t + x\partial_x + y\partial_y + z\partial_z\} \tag{1.1}$$

generates a nonbarochronic regular partially invariant solution (RPIS) of rank 1 and defect 1 [2]. It is conveniently written in cylindrical coordinates (x, r, θ), $\boldsymbol{u} = (u, V, W)$, where $r = \sqrt{y^2 + z^2}$, $\theta = \arctan z/y$, and u, V, and W are the axial (along the Ox axis), radial, and circumferential velocity components, respectively.

The invariants of algebra (1.1) are the following thermodynamic quantities: the density (ρ), the pressure (p), the entropy (S), the polar angle (θ), and the velocity components (V and W). The component u is a superfluous function. According to the general scheme of constructing RPIS [3], the solution generated by subalgebra (1.1) is represented as

$$V = V(\theta), \quad W = W(\theta), \quad \rho = \rho(\theta), \quad p = p(\theta), \quad S = S(\theta), \quad u = u(t, x, \theta, r). \tag{1.2}$$

Below, for definiteness, we consider the polytropic gas equation of state $p = S\rho^\gamma$ with the adiabatic exponent $\gamma > 1$, for which the sound velocity c is represented as $c^2 = \gamma p/\rho$. Substitution of representation (1.2) into the system of EGD yields a factor system for algebra (1.1) which consists of an invariant subsystem and an overdetermined subsystems. The compatibility conditions for the latter supplement the invariant subsystem. For the RPIS (1.2), this procedure is described in [2]; therefore, we shall give only the final result.

The velocity component u (superfluous function) has the form

$$u = hx/r + U(t, r, \theta), \tag{1.3}$$

where $h = h(\theta)$ is an auxiliary function which is a peculiar generalized potential of the solution. The function U in (1.3) is a solution of the equation

$$U_t + VU_r + (1/r)WU_\theta + (1/r)hU = 0. \tag{1.4}$$

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The invariant subsystem has the form

$$\begin{gathered}V' = W,\\ W(c^2)' + (\gamma - 1)c^2(W' + V + h) = 0,\\ Wh' + h^2 - hV = 0,\\ V^2 + W^2 + 2c^2/(\gamma - 1) = 2b_0,\end{gathered} \tag{1.5}$$

where the prime denotes the derivative with respect to the invariant variable θ; $b_0 > 0$ is a constant. In (1.5), the third equation is the compatibility condition of the overdetermined subsystem. System (1.5) consists of three differential equations for four unknown functions V, W, c, and h and a finite relation which is an invariant Bernoulli integral.

We note that by adding the term $u^2/2$ to both sides of the last equation of system (1.5), one obtains a complete Bernoulli integral. By virtue of the first momentum equation, $Du = 0$; therefore, on the right side of the obtained equality, the constant b_0 is replaced by a certain Lagrangian function of the coordinate u.

After the resolution of the invariant subsystem (1.5), Eq. (1.4) is integrated in quadratures. In [2], system (1.5) is reduced to an implicit ordinary differential equation of the third order by straightening the invariant part of the total derivative using the replacement $\theta \to \lambda = \lambda(\theta)$ according to the formula $d\lambda/d\theta = 1/W$ for $W \neq 0$. The resulting key equation of the third order is written in terms of the function h and its derivatives. In the study of the RPIS (1.2), the main point is to examine the solution of the dynamic system (1.5) or the equivalent equation for $g(\lambda)$ from [2].

The present paper gives the first stage of a qualitative analysis of the solution of system (1.5).

Representation (1.3) for the velocity component u implies that for $h \neq 0$, the gas motion is substantially three-dimensional. For $h = 0$, system (1.5) reduces to the well-known system that describes steady-state irrotational plane gas flows and has solutions in the form of simple Prandtl–Meyer waves [4]. Thus, the presence of the function $h \neq 0$ generates a new solution that generalizes the classical simple waves. For solution (1.2) in cylindrical coordinates, the vorticity vector $\boldsymbol{\omega} = (\omega^1, \omega^2, \omega^3)$ is represented as

$$\omega^1 = r^{-1}u_\theta, \qquad \omega^2 = -r^{-1}u_r, \qquad \omega^3 = r^{-1}((rW)_r - V_0) = 0.$$

Hence, by virtue of the first equation in (1.5), this is irrotational gas flow in the plane $R^2(y, z)$.

The presence of the function h considerably complicates the analysis of the equations and leads to a new physically meaningful solution.

2. Transformation of System (1.5). The last equation of system (1.5) is a finite integral; therefore, it is convenient to treat it as a dynamic system on the sphere S^2 in the space $R^3(V, W, c)$ or on the cylinder $R \times S^2$ in the space $R^4(h, V, W, c)$ and to pass to the surface specified by this integral using stereographic projection. First, the unknown functions are scaled as follows:

$$V = \sqrt{2b_0}\, v, \quad W = \sqrt{2b_0}\, w, \quad c = \sqrt{b_0(\gamma - 1)}\, Q, \quad h = \sqrt{2b_0}\, H, \quad Q > 0. \tag{2.1}$$

In the new variables (v, w, Q, and H) defined by (2.1), system (1.5) becomes

$$\begin{gathered}v' = w,\\ wQ' + \alpha_0^2 Q(w' + v + H) = 0,\\ wH' - vH + H^2 = 0,\\ v^2 + w^2 + Q^2 = 1, \qquad Q > 0,\end{gathered} \tag{2.2}$$

where $\alpha_0^2 = (\gamma - 1)/2$.

The stereographic projection maps the point P of the upper hemisphere $S^2(v, w, Q)$, $Q > 0$ to the point P' on the plane $R^2(R, \psi)$ according to the formulas [5]

$$v = \frac{2R\sin\psi}{R^2 + 1}, \qquad w = \frac{2R\cos\psi}{R^2 + 1}, \qquad Q = \frac{R^2 - 1}{R^2 + 1}. \tag{2.3}$$

The variables (R, ψ) are polar coordinates on the plane $R \geq 0$, $\psi \in [0, 2\pi)$. Because $Q > 0$ (this is the scaled sound velocity), it follows that $R > 1$, and, hence, on the plane $R^2(R, \psi)$ we consider only the exterior of the unit circle. The image of the north pole N of the hemisphere is an infinite point on the plane.

In the variables (R, ψ, H), system (2.2) is written as

$$R' = \frac{H(R^2-1)(R^2+1)^2\cos\psi}{2d},$$
$$\psi' = \frac{2Rd + H(R^2-1)^2(R^2+1)\sin\psi}{2Rd}, \tag{2.4}$$
$$H' = \frac{H(2R\sin\psi - (R^2+1))}{2R\cos\psi},$$

where $d = (R^2-1)^2 - \beta_0^2 R^2\cos^2\psi$ and $\beta_0 = 4/\alpha_0^2$ $(R > 1)$.

After the introduction of modified time and the new variable $g = \sin\psi$, system (2.4) reduces to a dynamic system with polynomial right sides. In the theory of dynamic systems, the independent variable is usually called time, although in our case, the independent variable is the polar angle θ. Modified time — the new independent variable τ — is introduced by the rule

$$2R\,|d\cos\psi|\,\frac{d}{d\theta} = \frac{d}{d\tau}. \tag{2.5}$$

The modulus sign in formula (2.5) ensures the monotonicity of the replacement. The zeroes of the expression $d\cos\psi$ can specify nonextendability manifolds of the solution because at them there is a change in the direction of motion on the trajectories of system (2.4) in the phase space (we return to this question in Sec. 6, which studies the behavior of the system on the boundaries of the domain of existence of the solution).

After these replacements, system (2.4) becomes

$$R' = HR(R^2-1)(R^2+1)^2(1-g^2),$$
$$g' = (1-g^2)(2Rd + H(R^2-1)^2(R^2+1)g), \tag{2.6}$$
$$H' = H(2Rg - H(R^2+1))d,$$

where

$$d = (R^2-1)^2 - \beta_0^2R^2(1-g^2). \tag{2.7}$$

System (2.6) is defined in the domain

$$\Omega: \quad R > 1, \quad |g| \leqslant 1. \tag{2.8}$$

The boundaries $\partial\Omega = \{\partial\Omega_1 = \{R = 1\}\bigcup \partial\Omega_2 = \{|g| = 1\}\}$ of the domain Ω are surfaces in the phase space $R^3(R, g, H)$. They are called the physical boundaries of the manifold on which system (2.6) is defined, in contrast to those components of the manifold boundary which are obtained by compactification at infinity.

3. General Information from the Theory of Dynamic Systems. The study of the qualitative properties of solutions of multidimensional dynamic systems has features not available in the two-dimensional case. For completeness of the discussion, we give the basic propositions of the theory [5–7].

We consider the dynamic system

$$\frac{dx_i}{dt} = f_i(x_1, \ldots, x_n), \qquad i = 1, \ldots, n. \tag{3.1}$$

In studying this system, the first stage is to find and classify the singular points $\boldsymbol{x}_0$ of system (3.1) that satisfy the equations

$$f_i(\boldsymbol{x}_0) = 0, \qquad i = 1, \ldots, n. \tag{3.2}$$

Let $J = (\partial f_i/\partial x_j)$ $(i, j = 1, \ldots, n)$ be the Jacobi matrix of the vector field of system (3.1), which defines the linearization of this system. The nonzero eigenvalues of the matrix J correspond to nondegenerate singular points. The eigenvalues $\lambda_1, \ldots, \lambda_d$ with negative real parts correspond to the d-dimensional manifold W_s^d on which all trajectories of system (3.1) enter a singular point $\boldsymbol{x}_0$ as $t \to +\infty$; for the eigenvalues $\lambda_1, \ldots, \lambda_k$ with positive real parts there exists a k-dimensional invariant manifold W_u^k on which all trajectories leave this point. Asymptotic formulas for the trajectories near the nondegenerate singular point are known.

The existence of a manifold M^k of dimension k which is entirely filled by singular points — a singular manifold — is possible. At the points M^k, system (3.1) necessarily has k zero eigenvalues. The singular manifold is called nondegenerate if at almost all singular points M^k there are $n-k$ eigenvalues with nonzero real parts.

TABLE 1

No.	Description of Manifold	Eigenvalues	Additional characteristic
1	Straight lines $l_\varepsilon = \{R = 1,\ u = \varepsilon,\ H \in R\}$	$\lambda_i = 0,\ i = 1, 2, 3$ (degenerate)	—
2	Straight lines $m_\varepsilon = \{R > 1,\ u = \varepsilon,\ H = 0\}$	$\lambda_1 = 0$ $\lambda_2 = -4\varepsilon R(R^2 - 1)^2$ $\lambda_3 = 2\varepsilon R(R^2 - 1)^2$ (nondegenerate for $R \neq 1$)	Degeneration at the points B_ε: $R = 1$
3	Curves $c_\varepsilon = \{R > 1,\ H = 2\varepsilon R/(R^2 + 1),\ u = \varepsilon\}$ In c_1: $0 < H < 1$ In c_{-1}: $-1 < H < 0$	$\lambda_1 = 0$ $\lambda_2 = -8\varepsilon R(R^2 - 1)^2$ $\lambda_3 = -2\varepsilon R(R^2 - 1)^2$ (nondegenerate for $R \neq 1$)	Degeneration at the points A_ε: $R = 1$
4	Points $B_\varepsilon = \{R = 1,\ u = \varepsilon,\ H = 0\}$	$\lambda_i = 0,\ i = 1, 2, 3$ (degenerate)	—
5	Points $A_\varepsilon = \{R = 1,\ H = \varepsilon,\ u = \varepsilon\}$	$\lambda_i = 0,\ i = 1, 2, 3$ (degenerate)	—

The degenerate singular points (all $\lambda_i = 0$) are resolved using a method for the resolution of singularities (blowing of the phase space at a singular point) in which a singular point is replaced by an invariant manifold. It is known that in a finite number of steps, this process leads to a nondegenerate singular point.

After finding all finite singular points and manifolds and resolving those that are, we construct the compact manifold on which system (3.1) is defined. Some components of the boundary of this manifold are defined by the physical conditions of the problem:

$$\Gamma_j: \quad \Phi_j = 0, \quad j = 1, \ldots, N. \tag{3.3}$$

In the examined case, conditions (3.3) are specified by the system of inequalities (2.8). In addition, system (3.1) is supplemented by the boundary at infinity for the variables x_i (an analog of the Poincaré sphere for two-dimensional dynamic systems) by introducing $2n$ maps $U_i^{\pm}$ of the projective coordinates

$$y_1^i = \frac{x_1}{x_i}, \quad \ldots, \quad y_j^i = \frac{x_1}{x_i}, \quad z_i = \frac{1}{x_i}, \quad \ldots, \quad y_n^i = \frac{x_n}{x_i}. \tag{3.4}$$

In the local map $U_i^{\pm}$, we have $z_i = \pm 1$. In coordinates (3.4), the infinite points x_i correspond to the points of the hyperplane $L_i^{\pm}$: $z_i = \pm 0$. As a result of transformation of coordinates of the form (3.4), a boundary — the $(n-1)$-dimensional sphere S^{n-1} covered by hyperplanes $L_i^{\pm}$ — is attached to the phase space in (2.2) in the region of infinitely large values of the coordinates x_i. This system should be studied on the invariant hyperplane L_i because new singular points can appear on it.

After these procedures, we obtain a dynamic system on the compact manifold S with all singular points and manifolds of the system being resolved, i.e., made nondegenerate.

The behavior of the trajectories of the dynamic system is studied by means of separatrix approximation. A sequence of separatrices passing from one singular point to another is constructed, and the behavior of the trajectories of the dynamic system (3.1) that close to these separatrices is examined using the theorem on continuous dependence of solutions on the initial data.

In the present paper, we find and classify all singular points and manifolds of system (3.1) and construct the compact manifold S on which this system is defined.

4. Finite Singular Points and Manifolds of System (2.6). The singular points and manifolds and the linearization eigenvalues of the vector field of system (2.6) at these points are calculated by a standard but rather tedious procedure. The final result of the classification of the finite singular points and manifolds of system (2.6) is given in Table 1. The points of intersection of the straight lines m_ε and curves c_ε with the plane $R = 1$ are degenerate singular points, on which the degree of regularity decreases. Everywhere in Table 1, $\varepsilon = \pm 1$.

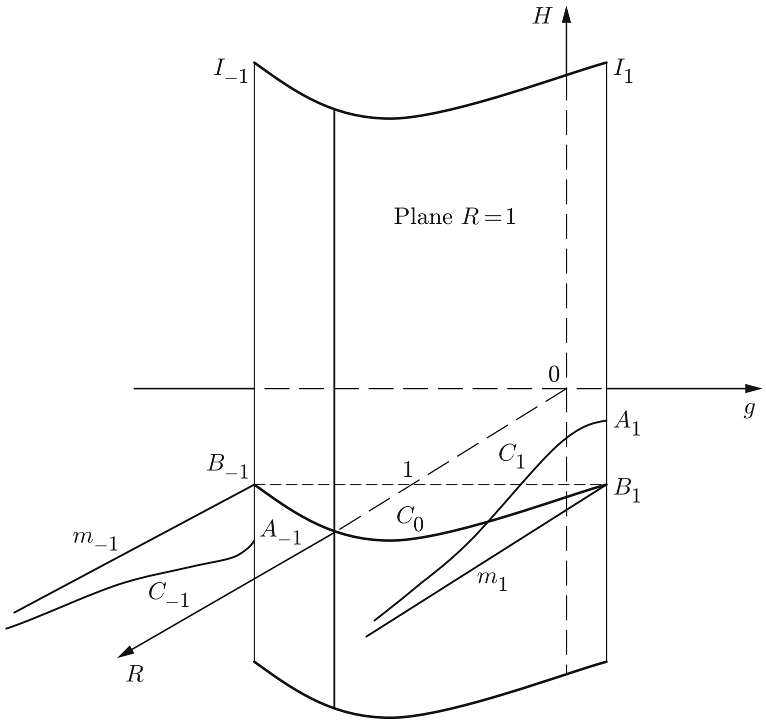

Fig. 1

Figure 1 shows the domain Ω (2.8) in which system (2.6) is defined; the singular manifolds and points indicated according to Table 1.

5. Singular Points of System (2.6) at Infinity and Its Compactification. To study the singular points of system (2.6) at infinity for the variables R and H, it is necessary to transform it to projective coordinates of the form (3.4). To cover the entire manifold, one needs several maps of the form of (3.4): one for $R = +\infty$ and two for $H = +\infty$ and $H = -\infty$, respectively. We perform the necessary calculations for one of them and then give the final result.

Let us find the singular points of system (2.6) for $R \to +\infty$ (we recall that $R > 1$). For this, we introduce new coordinates (R_0, H_0, and g) of the form (3.4) according to the formulas

$$R_0 = \frac{1}{R}, \quad H_0 = \frac{H}{R}, \quad g. \tag{5.1}$$

Because $R > 1$, it follows that $R_0 < 1$. After the monotonic replacement of time $R_0^7(d/d\tau) \to (d/d\tau)$, system (2.6) in the coordinates (5.1) becomes

$$\begin{gathered} R' = H_0 R_0 (R_0^2 - 1)(R_0^2 + 1)^2 (1 - g^2), \\ g' = (1 - g^2)(2R_0^2 d_0 + H_0(1 - R_0^2)^2 (R_0^2 + 1) g), \\ H' = H_0(2R_0 g - H_0(R_0^2 + 1) d_0 + H_0(R_0^2 - 1)(R_0^2 + 1)^2 (1 - g^2)), \end{gathered} \tag{5.2}$$

where $d_0 = (1 - R_0^2)^2 - \beta_0^2 R_0^2 (1 - g^2)$.

We find the singular points of system (5.2) for $R_0 = 0$ that correspond to $R = +\infty$. On the plane $R_0 = 0$, system (5.2) reduces to the system

$$\begin{gathered} g' = g H_0 (1 - g^2), \\ H_0' = H_0^2 (g^2 - 2), \end{gathered} \tag{5.3}$$

which after the modification of time $|H_0|^{-1}\, d/d\tau \to d/d\tau$ becomes

$$\begin{gathered} g' = g(1 - g^2), \\ H_0' = H_0 (g^2 - 2). \end{gathered} \tag{5.4}$$

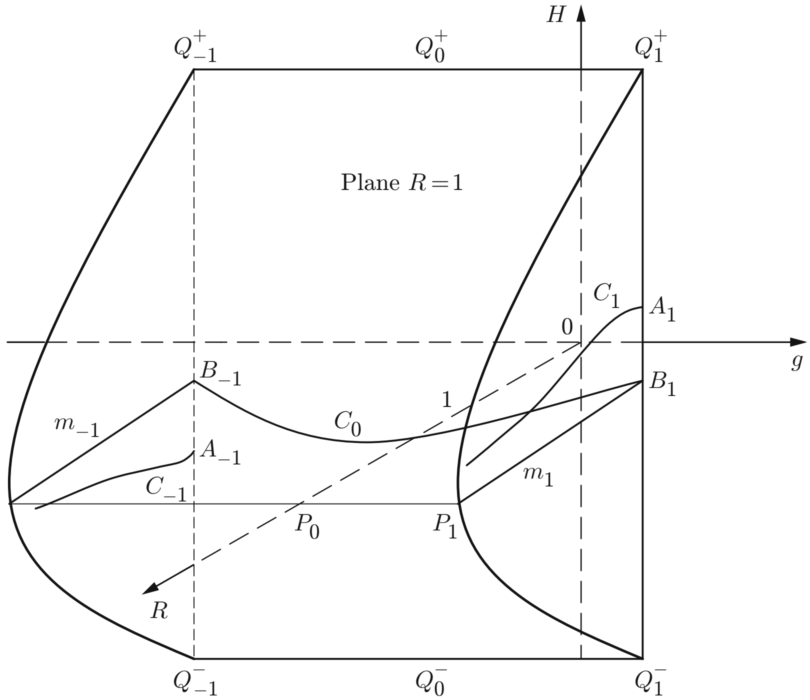

Fig. 2

TABLE 2

No.	Boundary components	Flow parameters	State of Gas
1	Plane $R=1$	$Q=0$	Rarefaction of gas Vacuum
2	Plane $g=\varepsilon$ $(\varepsilon=\pm 1)$	$v=\dfrac{2\varepsilon R}{R^2+1}$ $w=0$ $Q=\dfrac{R^2-1}{R^2+1}$	Radial gas motion ($\varepsilon=+1$ refers to a source and $\varepsilon=-1$ refers to a sink)
3	$R=+\infty$	$v=w=0$ $Q=1$	Stop of gas Rest
4	Plane $H=0$	Solutions of system for $h=0$	Prandtl–Meyer simple wave

System (5.4) has the following singular points: $P_0=\{u=H_0=0\}$ and $P_\varepsilon=\{u=\varepsilon,\ H_0=0\}$. At the point P_0, the linearization eigenvalues of the vector field of system (5.4) are equal to $\lambda_1=1$ and $\lambda_2=-2$, so that this point is a saddle. The points P_ε ($\lambda_1=-2$ and $\lambda_2=-1$) are attracting nodes.

The behavior of system (2.6) for $H\to\infty$ is studied similarly for two different cases: $H\to+\infty$ and $H\to-\infty$. For this, we introduce the coordinates $R_1=R/H$ and $H_1=1/H, g$, so that $H=\pm\infty$ corresponds to $H_1=\pm 0$. The result of the study is as follows. At the points $Q_0^{\pm}=\{R_1=u=0,\ H_1=\pm 0\}$, we have $\lambda_1=\lambda_2=0$. The resolution of these degenerate points shows that they are nodes. At the points $Q_\varepsilon^{\pm}=\{R_1=0,\ u=\varepsilon,\ H_1=\pm 0\}$, the eigenvalues are $\lambda_1=0$ and $\lambda_2=\varepsilon$, so that these points are saddles.

Figure 2 shows the compact manifold S of system (2.6) with the indication of the singular manifolds according to Table 1 and the singular points at infinity for the variables R and H.

The above classification of the singular points and manifolds of system (2.6) leads to the conclusion that all of them are on the components of the boundary of the manifold S; therefore, it is necessary to examine whether they correspond to any physically meaningful solutions.

6. Behavior of the Solutions of System (2.6) on the Components of the Boundary of the Compact Manifold S. Table 2 gives a description of the asymptotic modes of gas motion that correspond to the components of the boundary of S. All components of the boundary are invariant manifolds. The surface $d = 0$ specified in the phase space by Eq. (2.7) is the nonextendability surface of the solution: according to Eqs. (2.4), the vector field of the system has opposite directions on two sides of this surface. This surface is the image of the acoustic characteristic on the solution (1.2) in the phase space.

Indeed, in the physical space, it corresponds to a certain plane $\theta = \theta_0 = \text{const}$, and the velocity component normal to this is W. In the variables g and R, the equation $W^2 = c^2$ becomes $d = 0$, as this proves the statement.

The classical Prandtl–Meyer simple wave [4] corresponds to the invariant manifold in the solution of system (2.6) — the plane $H = 0$.

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